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The effect of solar wind on the charged particles' diffusion coefficients

J. F. WANG[1] AND G. QIN[1,2]

[1]*School of Science, Harbin Institute of Technology, Shenzhen, 518055, China; qingang@hit.edu.cn*

[2]*Shenzhen Key Laboratory of Numerical Prediction for Space Storm, Harbin Institute of Technology, Shenzhen, 518055, China*

ABSTRACT

The transport of energetic charged particles through magnetized plasmas is ubiquitous in interplanetary space and astrophysics, and the important physical quantities are the along-field and cross-field spatial diffusion coefficients of energetic charged particles. In this paper, the influence of solar wind on particle transport is investigated. Using the focusing equation, we obtain along- and cross-field diffusion coefficient accounting for the solar wind effect. For different conditions, the relative importance of solar wind effect to diffusion are investigated. It is shown that when energetic charged particles are close to the sun, for along-field diffusion the solar wind effect needs to be taken into account. These results are important for studying energetic charged particle transport processes in the vicinity of the sun.

Keywords: Interplanetary turbulence (830); Magnetic fields (994); Solar energetic particles (1491)

1. INTRODUCTION

The charged energetic particles emitted by the sun, known as solar energetic particles (SEPs), have crucial impacts on the environment of interplanetary and planetary space (Schlickeiser 2002; Reames 2017). For

Corresponding author: G. Qin
qingang@hit.edu.cn

example, SEPs can cause geomagnetic and ionospheric storms on the Earth, and even pose a threat to the safe operation of the ground power systems and cause leaks in underground oil pipelines. In addition, these energetic particles can reduce the reliability of spacecraft-borne detectors and endanger the health of astraurants and aircrew (Lanzerotti 2017; Mertens et al. 2018). The turbulent magnetized plasmas in the interplanetary space, e.g., the solar wind, have a significant impact on the transport of solar energetic particles. Therefore, it is extremely important to investigate the propagation of solar energetic particles through the solar wind (Jokipii 1966; Zhang 1999; Schlickeiser 2002; Matthaeus et al. 2003; Qin 2007; Schlickeiser et al 2007; Schlickeiser & Shalchi 2008; Shalchi 2010; Qin & Zhang 2014; Wang & Qin 2015; Zhang & Zhao 2017; Zhang et al. 2019; Zhao et al. 2017).

Due to the interaction with the background and superposed turbulent magnetic fields, the motion of charged energetic particles can be modelled as two components of motions, i.e., the helical motion around the mean magnetic field lines and the superposed stochastic one (Schlickeiser 2002; Shalchi 2009). Therefore, various statistical methods have to be utilized in related studies (Jokipii 1966; Schlickeiser 2002; Matthaeus et al. 2003; Shalchi 2009, 2010, 2020b). The well-known Master equation in statistical physics, which provides the most fundamental description of the transport of charged energetic particles through magnetized plasmas, is too complicated to be used in analytical investigations (Schlickeiser 2002; Shalchi 2009). Therefore, in the previous papers the relatively simple equations, such as the Fokker-Planck equation, have been widely used in plasma physics, astrophysics and space physics. The focusing equation, the special version of the Fokker-Planck equation, has been extensively employed in the research of the energetic particle transport in the heliosphere and the magnetosphere (Skilling 1971; Schlickeiser 2002; Qin et al. 2005, 2006; Zhang et al. 2009; Dröge et al. 2010; Zuo et al. 2011; Wang et al. 2012; Qin et al. 2013; Zuo et al. 2013; Wang et al. 2014; Zhang et al. 2019; Qin & Qi 2020).

The focusing equation includes all the important transport effects of SEPs in solar wind plasmas, e.g., the pitch-angle diffusion, the cross-field diffusion, the spatial convection, the adiabatic cooling, the adiabatic focusing, and so on, among which, the spatial convection, the adiabatic cooling, and the adiabatic focusing are affected by solar wind velocity effects (Skilling 1971; Schlickeiser 2002; Forbes et al. 2006; Shalchi 2009; Zhang & Zhao 2017; Wijsen et al. 2019; Zhang et al. 2019; Bian & Emslie 2020). These

effects in the focusing equation are not isolated from each other, but have mutual influence. The along- and cross-field spatial diffusion are very important transport processes of energetic charged particles, so they are widely studied in plasma physics (Schlickeiser 2002; Qin 2007; Shalchi 2009; Qin & Zhang 2014; Qin & Shalchi 2014; Shalchi 2020b). In addition, the impacts of along-field adiabatic focusing effect on the along- and cross-field diffusion have been extensively studied (Roelof 1969; Earl 1976; Kunstmann 1979; Beeck & Wibberenz 1986; Bieber & Burger 1990; Kota 2000; Schlickeiser & Shalchi 2008; Shalchi 2011; Litvinenko 2012a,b; Shalchi & Danos 2013; He & Schlickeiser 2014; Wang & Qin 2016; Wang et al. 2017; Wang & Qin 2018, 2019).

For the along-field diffusion, three different definitions of diffusive coefficient have been proposed in the past decades, i.e., the displacement variance definition

$$\kappa_{zz}^{DV} = \frac{1}{2} \lim_{t \to t_\infty} \frac{\mathrm{d}\sigma^2}{\mathrm{d}t} \tag{1}$$

with the first- and second-order moments of charged particle distribution function, the Fick's law definition

$$\kappa_{zz}^{FL} = \frac{J}{X} \tag{2}$$

with $X = \partial F / \partial z$, and the TGK formula definition

$$\kappa_{zz}^{TGK} = \int_0^\infty dt \langle v_z(t) v_z(0) \rangle. \tag{3}$$

If the mean magnetic field is uneven along the field lines, it has been proved that different definitions of along-field diffusion coefficient are not equivalent to each other, i.e., κ_{zz}^{DV}, rather than κ_{zz}^{FL} and κ_{zz}^{TGK}, is the most appropriate definition (Wang & Qin 2018, 2019). In addition, it is demonstrated that the cross-field diffusion coefficient $\kappa_\perp$ is modified by along-field non-uniformity of the mean magnetic field (Wang et al. 2017). Moreover, the influence of along-field adiabatic focusing on the momentum transport have also been investigated (Schlickeiser & Shalchi 2008; Litvinenko & Schlickeiser 2011; Wang & Qin 2021).

Cross-field diffusion, i.e., perpendicular diffusion, is another crucial transport process for charged particles in both space physics and laboratory plasma physics. Cross-field diffusion coefficient, which is the important parameter describing perpendicular transport, has been widely investigated in previous studies (Matthaeus et al. 2003; Shalchi 2010; Qin & Zhang 2014; Shalchi 2017, 2019, 2020a,b, 2021a,b, 2022). A

large number of studies have demonstated that along-field diffusion has a strong influence on perpendicular transport (Matthaeus et al. 2003; Shalchi et al. 2004, 2006; Shalchi 2008, 2010, 2017, 2018, 2019, 2020a,b, 2021a). In addition, along-field adiabatic focusing effect is another factor influencing cross-field diffusion (Wang et al. 2017). However, the effect of solar wind on cross-field diffusion has not been investigated in the previous paper. In this paper, we also explore this problem.

The remainder of this paper is organized as follows. In Section 2, the focusing equation that satisfies particle number conservation is introduced. In Section 3, the along- and cross-field diffusion coefficients of energetic charged particles including solar wind effects are derived. In Section 4, the influence of the solar wind effect on along-field diffusion is explored, with the dimensionless quantities determining the relative importance of the solar wind effect to diffusion transport derived. In Section 5, the effect of solar wind on cross-field diffusion is expored. We conclude and summarize our results in Section 6.

2. THE FOCUSING EQUATION

The Fokker-Planck equation is formulated as follows (Duderstadt & Martin 1979; Huang & Ding 2008; Zank 2014)

$$
\frac{\partial f}{\partial t} + \nabla \cdot (\boldsymbol{u} f) + \nabla_p \cdot (\boldsymbol{a} f) = \nabla \cdot (\kappa_1 \cdot \nabla f) + \nabla_p \cdot \left(\kappa_1^p \cdot \nabla_p f \right) + \nabla \cdot [\kappa_{21} \cdot \nabla (\kappa_{22} \cdot \nabla f)]
$$
$$
+ \nabla_p \cdot \left[\kappa_{21}^p \cdot \nabla_p \left(\kappa_{22}^p \cdot \nabla_p f \right) \right] + \cdots . \tag{4}
$$

Here, $f = f(\boldsymbol{r}, \boldsymbol{p}, t)$ is the distribution function of charged energetic particles, t is time, $\boldsymbol{u}$ is particle velocity, and $\boldsymbol{a}$ is particle acceleration. In addition, the operators ∇ and ∇_p are the spatial and momentum Laplacians, respectively. For simplicity, only the first- and second-order derivative terms of the Fokker-Planck equation are usually retained (Huang & Ding 2008), thus Equation (4) becomes

$$
\frac{\partial f}{\partial t} + \nabla \cdot (\boldsymbol{u} f) + \nabla_p \cdot (\boldsymbol{a} f) = \nabla \cdot (\kappa \cdot \nabla f) + \nabla_p \cdot \left(\kappa_p \cdot \nabla_p f \right) . \tag{5}
$$

In general, the terms of the first- and second-order derivative in Equation (5) can describe most of the important specific physical effects of energetic particle transport in solar wind plasmas, e.g., pitch-angle scattering, cross-field diffusion, along-field adiabatic focusing, adiabatic cooling, along-field spatial convection, etc. To encompasses all of the aforementioned physical processes, the focusing equation becomes

the Fokker-Planck equation,

$$
\begin{aligned}
\frac{\partial f}{\partial t} &= \nabla \cdot (\kappa_\perp \cdot \nabla f) - \nabla \cdot \left[\left(u\mu\hat{b} + V \right) f \right] + \frac{\partial}{\partial \mu} \left(D_{\mu\mu} \frac{\partial f}{\partial \mu} \right) \\
&+ \frac{1}{p^2} \frac{\partial}{\partial p} \left\{ p^3 \left[\frac{1 - \mu^2}{2} \left(\nabla \cdot V - \hat{b}\hat{b} : \nabla V \right) + \mu^2 \hat{b}\hat{b} : \nabla V \right] f \right\} \\
&+ \frac{\partial}{\partial \mu} \left\{ \frac{1 - \mu^2}{2} \left[-\frac{u}{L} - \mu \left(\nabla \cdot V - 3\hat{b}\hat{b} : \nabla V \right) \right] f \right\},
\end{aligned}
\tag{6}
$$

where $\kappa_\perp$ is the perpendicular diffusion coefficient tensor, u is the particle speed, μ is pitch-angle cosine, $\hat{b}$ is the unit vector along the background magnetic field, p is the magnitude of momentum, $D_{\mu\mu}(\mu)$ is the pitch-angle diffusion coefficient, $L = \left(\hat{b} \cdot \nabla \ln B \right)^{-1}$ with solar mean magneitc field B is the characteristic length of the adiabatic focusing, and V is the solar wind velocity. The latter equation satisfies the conservation law of particle number, and the detailed derivative is shown in Appendix A. For convenience, the focusing equation can be rewritten as follows

$$
\begin{aligned}
\frac{\partial f}{\partial t} &= \nabla \cdot (\kappa_\perp \cdot \nabla f) - \nabla \cdot \left[\left(u\mu\hat{b} + V \right) f \right] + \frac{\partial}{\partial \mu} \left(D_{\mu\mu} \frac{\partial f}{\partial \mu} \right) + \frac{\partial}{\partial \mu} \left[\frac{u \left(1 - \mu^2 \right)}{2L} f \right] \\
&+ \frac{1}{p^2} \frac{\partial}{\partial p} \left\{ p^3 \left[\frac{1 - \mu^2}{2} \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} \right) + \mu^2 \frac{\partial V_z}{\partial z} \right] f \right\} - \frac{\partial}{\partial \mu} \left[\mu \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} - 2\frac{\partial V_z}{\partial z} \right) f \right].
\end{aligned}
\tag{7}
$$

The derivation details from Equation (6) to (7) are shown in Appendix B.

3. THE DIFFUSION COEFFICIENTS INCLUDING SOLAR WIND EFFECTS

In this section, we explore the diffusion coefficients of energetic charged particles including solar wind effects.

3.1. *The along-field diffusion coefficient including solar wind effects*

Firstly, we derive the along-field diffusion coefficient formula of energetic particles including solar wind.

3.1.1. *The simplified Fokker-Planck equation*

For simplificty, the effect of solar wind spatial gradient are ignored. Moreover, by performing the integration $\int \mathrm{d}x \int \mathrm{d}y$ on Equation (7), we find

$$
\frac{\partial f_a}{\partial t} = -u\mu \frac{\partial f_a}{\partial z} - \frac{\partial}{\partial z} \left(\int \mathrm{d}x \int \mathrm{d}y V_z f \right) + \frac{\partial}{\partial \mu} \left[D_{\mu\mu} \frac{\partial f_a}{\partial \mu} - \frac{u \left(1 - \mu^2 \right)}{2L} f_a \right].
\tag{8}
$$

Here, $f_a(z, \mu, t) = \int dx \int dy f(x, y, z, \mu, t)$ is the distribution function. Now, we obtain the simplified Fokkerr-Planck equation.

3.1.2. The anisotropic distribution function $g_a(z, \mu, t)$

The distribution function of charged energetic particles can be divided into the isotropic part F_a, which satisfies

$$F_a = \frac{1}{2} \int_{-1}^{1} d\mu f_a, \tag{9}$$

and the anisotropic component, which satisfies the condition

$$\int_{-1}^{1} d\mu g_a = 0. \tag{10}$$

That is, the following formula holds

$$f_a = F_a + g_a, \tag{11}$$

By integrating Equation (8) over μ from -1 to 1, we can obtain

$$\frac{\partial F_a}{\partial t} = -\frac{u}{2} \frac{\partial}{\partial z} \int_{-1}^{1} d\mu \mu g_a - \frac{\partial}{\partial z} \left(\int dx \int dy V_z F \right). \tag{12}$$

Here, $F = \int_{-1}^{1} d\mu f/2$ is employed. In addition, the boundary condition $D_{\mu\mu}(\mu = \pm 1) = 0$ is also used. Similarly, integrating Equation (8) over μ from -1 to μ, we obtain

$$\frac{\partial F_a}{\partial t}(\mu + 1) + \frac{\partial}{\partial t} \int_{-1}^{\mu} dv g_a = -u \frac{\mu^2 - 1}{2} \frac{\partial F_a}{\partial z} - u \frac{\partial}{\partial z} \int_{-1}^{\mu} dv v g_a - (\mu + 1) \frac{\partial}{\partial z} \left(\int dx \int dy V_z F \right)$$

$$- \frac{\partial}{\partial z} \left(\int dx \int dy V_z \int_{-1}^{\mu} dv g(v) \right) + D_{\mu\mu} \frac{\partial g_a}{\partial \mu} - \frac{u\left(1 - \mu^2\right)}{2L} F_a - \frac{u\left(1 - \mu^2\right)}{2L} g_a \tag{13}$$

with $g(x, y, z, \mu, t) = f(x, y, z, \mu, t) - F(x, y, z, t)$. Equation (13) can be rewritten as

$$\frac{\partial g_a}{\partial \mu} + \frac{u\left(1 - \mu^2\right) g_a}{2D_{\mu\mu}L} + \frac{u\left(1 - \mu^2\right)}{2D_{\mu\mu}} \left(\frac{\partial F_a}{\partial z} + \frac{F_a}{L} \right) = \Phi_a(\mu) \tag{14}$$

with

$$\Phi_a(\mu) = \frac{1}{D_{\mu\mu}} \left[\frac{\partial F_a}{\partial t}(\mu + 1) + \frac{\partial}{\partial t} \int_{-1}^{\mu} dv g_a \right.$$

$$\left. + u \frac{\partial}{\partial z} \int_{-1}^{\mu} dv v g_a + (\mu + 1) \frac{\partial}{\partial z} \left(\int dx \int dy V_z F \right) + \frac{\partial}{\partial z} \left(\int dx \int dy V_z \int_{-1}^{\mu} dv g(v) \right) \right] \tag{15}$$

To continue, Equation (14) can be rewritten as (He & Schlickeiser 2014; Wang & Qin 2018, 2019)

$$\frac{\partial}{\partial \mu}\left\{\left[g_a(\mu) + L\left(\frac{\partial F_a}{\partial z} + \frac{F_a}{L}\right)\right]e^{-M(\mu)}\right\} = e^{-M(\mu)}\Phi_a(\mu) \tag{16}$$

with

$$M(\mu) = -\frac{u}{2L}\int_{-1}^{\mu} dv \frac{1 - v^2}{D_{vv}(v)}, \tag{17}$$

By integrating Equation (16) over v from -1 to μ, we can obtain the anisotropic distribution function as follows

$$g_a(\mu) = -L\left(\frac{\partial F_a}{\partial z} + \frac{F_a}{L}\right)\left[1 - \frac{2e^{M(\mu)}}{\int_{-1}^{1} d\mu e^{M(\mu)}}\right] + e^{M(\mu)}\left[R_a(\mu) - \frac{\int_{-1}^{1} d\mu e^{M(\mu)}R_a(\mu)}{\int_{-1}^{1} d\mu e^{M(\mu)}}\right] \tag{18}$$

with

$$R_a(\mu) = \int_{-1}^{\mu} dv e^{-M(v)}\Phi_a(v). \tag{19}$$

3.1.3. The governing equation of the isotropic distribution function $F_a(z, t)$

In order to derive the governing equation of the isotropic distribution function $F_a(z, t)$, we have to deduce the following integral

$$\frac{u}{2}\frac{\partial}{\partial z}\int_{-1}^{1} d\mu \mu g_a = \left(u\frac{\partial F_a}{\partial z} + uL\frac{\partial^2 F_a}{\partial z^2}\right)\frac{\int_{-1}^{1} d\mu \mu e^{M(\mu)}}{\int_{-1}^{1} d\mu e^{M(\mu)}}$$

$$+ \frac{u}{2}\int_{-1}^{1} d\mu \mu e^{M(\mu)}\left[\frac{\partial}{\partial z}R_a(\mu) - \frac{1}{\int_{-1}^{1} d\mu e^{M(\mu)}}\int_{-1}^{1} d\mu e^{M(\mu)}\frac{\partial}{\partial z}R_a(\mu)\right]. \tag{20}$$

The latter formula shows that the derivative of $R_a(z, \mu, t)$ with respect to z has to be deduced

$$\frac{\partial}{\partial z}R_a(\mu) = \frac{\partial^2 F_a}{\partial z \partial t}\int_{-1}^{\mu} dv \frac{e^{-M(v)}}{D_{vv}}(v + 1) + \int_{-1}^{\mu} dv \frac{e^{-M(v)}}{D_{vv}}(v + 1)\frac{\partial^2}{\partial z^2}\left(\int dx \int dy V_z F\right)$$

$$+ \int_{-1}^{\mu} dv \frac{e^{-M(v)}}{D_{vv}}\frac{\partial^2}{\partial t \partial z}\int_{-1}^{v} d\rho \int dx \int dy V_z F + u\int_{-1}^{\mu} dv \frac{e^{-M(v)}}{D_{vv}}\frac{\partial^2}{\partial z^2}\int_{-1}^{v} d\rho \rho g_a$$

$$+ \int_{-1}^{\mu} dv \frac{e^{-M(v)}}{D_{vv}}\frac{\partial^2}{\partial z^2}\left(\int dx \int dy V_z \int_{-1}^{v} d\rho g_a\right). \tag{21}$$

With Equations (20) and (21), the governing equation of the isotropic distribution function can be found

$$\frac{\partial F_a}{\partial t} = -\frac{\partial}{\partial z}\left(\int dx \int dy V_z F_a\right) - \kappa_z \frac{\partial F_a}{\partial z} + \kappa_{zz}\frac{\partial^2 F_a}{\partial z^2} + \kappa_{tz}\frac{\partial^2 F_a}{\partial t \partial z} + \kappa_{zz}^V \frac{\partial^2}{\partial z^2}\left(\int dx \int dy V_z F_a\right) \tag{22}$$

with and

$$\kappa_z = -u \frac{\int_{-1}^{1} d\mu \mu e^{M(\mu)}}{\int_{-1}^{1} d\mu e^{M(\mu)}}, \tag{23}$$

$$\kappa_{zz} = -uL \frac{\int_{-1}^{1} d\mu \mu e^{M(\mu)}}{\int_{-1}^{1} d\mu e^{M(\mu)}}$$

$$+ \frac{u^2}{2} \int_{-1}^{1} d\mu \mu e^{M(\mu)} \int_{-1}^{\mu} d\nu \frac{e^{-M(\nu)}}{D_{\nu\nu}} \int_{-1}^{\nu} d\rho \rho \left[1 - \frac{2e^{M(\rho)}}{\int_{-1}^{1} d\mu e^{M(\mu)}} \right]$$

$$- \frac{u^2}{2} \frac{\int_{-1}^{1} d\mu \mu e^{M(\mu)}}{\int_{-1}^{1} d\mu e^{M(\mu)}} \int_{-1}^{1} d\mu e^{M(\mu)} \int_{-1}^{\mu} d\nu \frac{e^{-M(\nu)}}{D_{\nu\nu}} \int_{-1}^{\nu} d\rho \rho \left[1 - \frac{2e^{M(\rho)}}{\int_{-1}^{1} d\mu e^{M(\mu)}} \right], \tag{24}$$

$$\kappa_{tz} = -\frac{u}{2} \int_{-1}^{1} d\mu \mu e^{M(\mu)} \int_{-1}^{\mu} d\nu e^{-M(\nu)} \frac{\nu + 1}{D_{\nu\nu}} + \frac{u}{2} \frac{\int_{-1}^{1} d\mu \mu e^{M(\mu)}}{\int_{-1}^{1} d\mu e^{M(\mu)}} \int_{-1}^{1} d\mu e^{M(\mu)} \int_{-1}^{\mu} d\nu e^{-M(\nu)} \frac{\nu + 1}{D_{\nu\nu}}$$

$$- \frac{u}{2} \int_{-1}^{1} d\mu \mu e^{M(\mu)} \int_{-1}^{\mu} d\nu \frac{e^{-M(\nu)}}{D_{\nu\nu}} \int_{-1}^{\nu} d\rho \left(\frac{2e^{M(\rho)}}{\int_{-1}^{1} d\mu e^{M(\mu)}} - 1 \right)$$

$$+ \frac{u}{2} \frac{\int_{-1}^{1} d\mu \mu e^{M(\mu)}}{\int_{-1}^{1} d\mu e^{M(\mu)}} \int_{-1}^{1} d\mu e^{M(\mu)} \int_{-1}^{\mu} d\nu \frac{e^{-M(\nu)}}{D_{\nu\nu}} \int_{-1}^{\nu} d\rho \left(\frac{2e^{M(\rho)}}{\int_{-1}^{1} d\mu e^{M(\mu)}} - 1 \right) \tag{25}$$

$$\kappa_{zz}^V = \frac{u}{2} \frac{\int_{-1}^{1} d\mu \mu e^{M(\mu)}}{\int_{-1}^{1} d\mu e^{M(\mu)}} \int_{-1}^{1} d\mu e^{M(\mu)} \int_{-1}^{\mu} d\nu \frac{e^{-M(\nu)}}{D_{\nu\nu}} (\nu + 1)$$

$$- \frac{u}{2} \int_{-1}^{1} d\mu \mu e^{M(\mu)} \int_{-1}^{\mu} d\nu \frac{e^{-M(\nu)}}{D_{\nu\nu}} (\nu + 1). \tag{26}$$

Here, only the terms containing the first- and second-order derivatives are retained. In fact, the higher-order derivative terms do not affect the results obtained in this article (Wang & Qin 2018, 2019).

3.1.4. *The along-field diffusion coefficient formula*

To derive the formula of mean square displacement definition, we have to obtain the first- and second-order moments of the isotropic distribution function, which are shown as follows

$$\frac{d}{dt} \langle z \rangle = \langle V_z \rangle + \kappa_z, \tag{27}$$

$$\frac{d}{dt} \langle z^2 \rangle = 2 \langle z V_z \rangle + 2\kappa_z \langle z \rangle + 2\kappa_{zz} - 2\kappa_{tz} \frac{d}{dt} \langle z \rangle + 2\kappa_{zz}^V \langle V_z \rangle. \tag{28}$$

Combining the latter formulas gives

$$\frac{1}{2} \frac{d\sigma^2}{dt} = \frac{1}{2} \frac{d}{dt} \left(\langle z^2 \rangle - \langle z \rangle^2 \right) = T_1 - T_2 + T_3 + T_4 \tag{29}$$

with

$$T_1 = \kappa_{zz}, \tag{30}$$

$$T_2 = \kappa_{tz}\kappa_z, \tag{31}$$

$$T_3 = \zeta = \langle zV_z \rangle - \langle z \rangle \langle V_z \rangle, \tag{32}$$

$$T_4 = \left(\kappa_{zz}^V - \kappa_{tz} \right) \langle V_z \rangle. \tag{33}$$

From Equations (32) and (33), we can find that the mean square displacement definition of along-field diffusion coefficient includes the solar wind and adiabatic focusing effects. According to the results obtained by Wang & Qin (2019), the term T_4 is approximately equal to zero. Thus, Equation (29) becomes

$$\frac{1}{2}\frac{d\sigma^2}{dt} = \frac{1}{2}\frac{d}{dt}\left(\langle z^2 \rangle - \langle z \rangle^2 \right) = T_1 - T_2 + T_3. \tag{34}$$

Note that if we only consider adiabatic focusing, Equation (34) becomes

$$\frac{1}{2}\frac{d\sigma^2}{dt} = T_1 - T_2 = \kappa_{zz} - \kappa_{tz}\kappa_z, \tag{35}$$

which is identical to the result derived by Wang & Qin (2018).

3.2. *The cross-field diffusion coefficient including solar wind effects*

Next, we derive the cross-field diffusion coefficient of energetic charged particles including solar wind effect.

3.2.1. *The govering equation of isotropic distribution function*

The starting point of the investigation in this subsection is also the focusing equation, which is displayed in Section 2. For Equation (7), by ignoring the terms containing spatial derivaitve of solar wind speed and integrating over μ from -1 to 1, we can obtain the governing equation of isotropic distribution function, which is given as follows

$$\frac{\partial F'}{\partial t} = \nabla \cdot (\kappa_\perp \cdot \nabla F') - \frac{u}{2}\frac{\partial}{\partial z}\int_{-1}^{1} d\mu\mu g - \nabla \cdot (VF') \tag{36}$$

with

$$F' = \frac{1}{2}\int_{-1}^{1} d\mu f. \tag{37}$$

Here, $g(x, y, z, \mu, t)$ is the anisotropic distribution function. Performing integrating $\int dy \int dz$ on Equation (36) yields

$$\frac{\partial F_b}{\partial t} = \frac{\partial}{\partial x}\left(\kappa_\perp \frac{\partial F_b}{\partial x}\right) - \frac{\partial}{\partial x}\left(\int dy \int dz V_x F'\right). \tag{38}$$

Here, $F_b = F(x, t)$ is the isotropic distribution function satisfying the following formula

$$F_b = \frac{1}{2}\int_{-1}^{1} d\mu \int dy \int dz f. \tag{39}$$

3.2.2. *The mean square displacement definition of perpendicular diffusion in x direction*

From Equation (38), the first- and second-order moments of charged particle distribution function can be obtained as

$$\frac{d}{dt}\langle x \rangle = \int dx \int dy \int dz V_x F' = \langle V_x \rangle, \tag{40}$$

$$\frac{d}{dt}\left\langle x^2 \right\rangle = 2\kappa_\perp + 2\int dx \int dy \int dz x V_x F' = 2\kappa_\perp + 2\langle x V_x \rangle. \tag{41}$$

Combining the latter formulas gives

$$\kappa_{xx}^{DV} = \frac{1}{2}\frac{d\sigma_x^2}{dt} = \frac{1}{2}\frac{d}{dt}\left(\langle x^2 \rangle - \langle x \rangle^2\right) = \kappa_\perp + \eta \tag{42}$$

with

$$\eta = \langle x V_x \rangle - \langle x \rangle \langle V_x \rangle. \tag{43}$$

Here, $\kappa_\perp = \kappa_{xx}^{FL}$ is the Fick's law definition. In addition, η is the solar wind effect on the cross-field diffusion. Thus, Equation (43) indicates that the cross-field diffusion is affected by solar wind effect.

In the following, we would discuss the influence of solar wind on along- and cross-field diffusion, i.e., Equations (32) and (43), with the typical parameter values, e.g., solar wind speed $V \sim 3 \times 10^5$ m/s, the solar rotation speed $\omega \sim 3 \times 10^{-6}$ /s, the energetic proton speed $u \sim 10^7$ m/s, the particle parallel mean free path $\lambda \sim 10^{10}$ m, and the parameter $\lambda_\perp \equiv 3\kappa_\perp/u = 10^9$ m.

4. THE INFLUENCE OF SOLAR WIND ON ALONG-FIELD DIFFUSION

In this section, based on Equation (34) we discuss the solar wind effect on along-field diffusion of energetic charged particles. If we only consider solar wind effect, Equation (34) is simplifies as

$$\kappa_{zz}^{DV} = \frac{1}{2} \frac{d\sigma^2}{dt} = T_1 + T_3 = \kappa_{zz} + \zeta, \tag{44}$$

with

$$\zeta = \langle zV_z \rangle - \langle z \rangle \langle V_z \rangle. \tag{45}$$

Here, $\sigma^2 = \langle z^2 \rangle - \langle z \rangle^2$, and ζ is the influence of solar wind on along-field diffusion coefficient. It is obvious that Equation (29) includes not only adiabatic focusing effect but also solar wind effect. Equations (44) and (45) show that the mean square displacement definition κ_{zz}^{DV} of energetic charged particles is not equal to the Fick' law definition κ_{zz}^{FL}. According to the results found by Wang & Qin (2019), with along-field adiabatic focusing, the mean square displacement definition κ_{zz}^{DV} is more appropriate than both Fick' law one κ_{zz}^{FL} and the Taylor-Green-Kubo one κ_{zz}^{TGK}. If the same operation performed as Wang & Qin (2019) for Equation (22), the similar conclusion can be easily obtained.

Equations (44) and (45) show that the mean square displacement definition κ_{zz}^{DV} accounts for the solar wind effect. Therefore, when the transport of SEPs parallel to the mean magnetic field is investigated, the relative importance of the solar wind effect to along-field diffusion has to be explored.

4.1. *The integrating form of the solar wind effect, ζ*

In order to investigate the relative importance of the solar wind effect on along-field diffusion, it is necessary to derive the specific form of the solar wind effect, ζ. Due to the magnetic freezing effect of plasmas and the solar rotation, the mean magnetic field of the sun presents a spiral form, which is called the Parker spiral. For simplicity, we consider it in the ecliptic plane. As shown in Figure 1, we consider a point A with the solar magnetic field line $\mathcal{L}_{\mathrm{MF}}$ going through it. We set a coordinate system $x' - y' - z'$, with the y'-axis pointing from the sun to point A, the z'-axis perpendicular to the ecliptic plane pointing towards the north, the x'-axis defined using the right-hand rule, and the origin O located at the center of the sun. We also set a polar coordinate system $r - \theta$, with the polar axis r tangent to $\mathcal{L}_{\mathrm{MF}}$ at point O, the polar angle θ defined in a clockwise direction. In addition, another coordinate system $x'' - y'' - z''$ is established by rotating $x' - y' - z'$

system with $\pi/2 - \theta_A$ clockwise through point O, where θ_A is the polar angle of point A. Additionly, we set a magnetic coordinate system $x - y - z$, with the z-axis along the tangent of magnetic field line towards its positive direction, the y-axis parallel to z', and the x-axis satisfying right-hand rule. Moreover, the tangent of $\mathcal{L}_{MF}$ at point A is the straight line $y' = kx' + r_0$ with the intercept r_0, i.e., the distance from O to A. We suppose the straight line can be written as $y'' = k'x'' + l$ in $x'' - y'' - z''$ system. Furthermore, normal equation of $\mathcal{L}_{MF}$ at point A is $y' = -x'/k + r_0$.

Additionally, the angle between z-axis and y'-axis is denoted as ψ. Therefore, the component of solar wind speed along solar magnetic field can be expressed as

$$V_z = V\cos\psi, \tag{46}$$

where V is the magnitude of the solar wind velocity $\mathbf{V}$. It is obvious that the angle ψ varies depending on the point on the magnetic field, and obeys the following formula on the ecliptic plane (e.g., Qin & Wu 2021)

$$\cos\psi = \left(1 + \frac{\omega^2 r^2}{V^2}\right)^{-1/2}. \tag{47}$$

Here, ω is the angular velocity of the sun, and r is the radial distance from the center of the sun to the point on solar magentic field. Thus, the influence ζ of the solar wind can be written as

$$\zeta = \int_{z_1}^{z_2} \mathrm{d}z F(z,t) V \left(1 + \frac{\omega^2 r^2}{V^2}\right)^{-1/2} - \int_{z_1}^{z_2} \mathrm{d}z z F(z,t) \int_{z_1}^{z_2} \mathrm{d}z F(z,t) V \left(1 + \frac{\omega^2 r^2}{V^2}\right)^{-1/2}, \tag{48}$$

where z_1 and z_2 are the integral lower and upper limits, respectively. The integrals in Equation (48) are along the curve of the mean magnetic field, and the normalization condition needs to be satisfied

$$\int_{z_1}^{z_2} \mathrm{d}F(z,t) = 1. \tag{49}$$

4.2. *Exploring the relative importance of solar wind effect to along-field diffusion*

In this paper, for the sake of simplicity, we assume that the speed V of the solar wind is constant. Thus, Equation (48) can be written as

$$\zeta = V \int_{z_1}^{z_2} \mathrm{d}z z F(z,t) \left(1 + \frac{\omega^2 r^2}{V^2}\right)^{-1/2} - V \int_{z_1}^{z_2} \mathrm{d}z z F(z,t) \int_{z_1}^{z_2} \mathrm{d}z F(z,t) \left(1 + \frac{\omega^2 r^2}{V^2}\right)^{-1/2}. \tag{50}$$

In coodinate system $x' - y' - z'$, if integral interval is small enought, the straight line $y' = kx' + r_0$ with $k = \tan\beta = \cot\psi$ can be used to approxmiately replace the curve of magnetic field line for the integrals in

Equation (50), and the distribution function $F(x', t)$ can be used to replace $F(z, t)$. Accordingly, as shown in Figure 1, the distance from O to any point, (x', y'), between the integral interval $[z_1, z_2]$ can be expressed as $r = \sqrt{x'^2 + (kx' + r_0)^2}$. In addition, we can obtain the formulas $dz = \sqrt{1 + (dy'/dx')^2}dx'$ and $z \approx \sqrt{1 + (dy'/dx')^2}x'$. To proceed, if the integral interval $[z_1, z_2]$ is set as $[-0.1r_0, 0.1r_0]$ in coordinate system $x' - y' - z'$, with the above setting, Equation (51) becomes

$$
\zeta = V\left(1 + k^2\right) \int_{-0.1r_0}^{0.1r_0} dx' \, x' F(x', t) \left[1 + \frac{\omega^2 \left[x'^2 + (kx' + r_0)^2\right]}{V^2}\right]^{-\frac{1}{2}}
$$
$$
- V\left(1 + k^2\right)^{3/2} \int_{-0.1r_0}^{0.1r_0} dx' \, x' F(x', t) \int_{-0.1r_0}^{0.1r_0} dx' F(x', t) \left[1 + \frac{\omega^2 \left[x'^2 + (kx' + r_0)^2\right]}{V^2}\right]^{-\frac{1}{2}}. \tag{51}
$$

For mathematical tractability, in this article, we only explore the tail of SEPs, for which the distribution function $F(x', t)$ is approximately uniform and is also approximately an even function of variable x' in integral interval $[-0.1r_0, 0.1r_0]$. Thus, the second term on the right-hand side of Equation (51) is equal to zero, and we have

$$
\zeta = V\left(1 + k^2\right) \int_{-0.1r_0}^{0.1r_0} dx' \, x' F(x', t) \left[1 + \frac{\omega^2 \left[x'^2 + (kx' + r_0)^2\right]}{V^2}\right]^{-\frac{1}{2}}. \tag{52}
$$

In fact, based on the characterisitcs of each SEP events, the integral interval can also be set to other values. In this paper, we only qualitatively explore the solar wind effect on energetic particle along-field diffusion. Therefore, the interval length in Equation (52) does not affect the findings obtained in this paper.

With the integral interval $[-0.1r_0, 0.1r_0]$ and Equation (49), the normalization condition becomes

$$
F(x', t) = \frac{5}{r_0}. \tag{53}
$$

Accordingly, the background solar wind effect on along-field diffusion can be written as

$$
\zeta = \left(1 + k^2\right) \frac{5V}{r_0} \int_{-0.1r_0}^{0.1r_0} dx' \, x' \left[1 + \frac{\omega^2 \left[x'^2 + (kx' + r_0)^2\right]}{V^2}\right]^{-\frac{1}{2}}. \tag{54}
$$

With integration by parts and a lengthy mathematical performance, the latter formula becomes

$$
\zeta = \frac{2k\frac{V^2}{\omega}}{\sqrt{\left(0.1 + \frac{1}{1+k^2}k\right)^2 (1 + k^2) + \frac{1}{(1+k^2)} + \frac{V^2}{\omega^2 r_0^2}} + \sqrt{\left(-0.1 + \frac{1}{1+k^2}k\right)^2 (1 + k^2) + \frac{1}{(1+k^2)} + \frac{V^2}{\omega^2 r_0^2}}}
$$

$$-\frac{5k}{\sqrt{1+k^2}}\frac{V^2}{\omega}\ln\frac{\left(0.1+\frac{1}{1+k^2}k\right)\sqrt{1+k^2}+\sqrt{\left(0.1+\frac{1}{1+k^2}k\right)^2(1+k^2)+\frac{1}{(1+k^2)}+\frac{V^2}{\omega^2 r_0^2}}}{\left(-0.1+\frac{1}{1+k^2}k\right)\sqrt{1+k^2}+\sqrt{\left(-0.1+\frac{1}{1+k^2}k\right)^2(1+k^2)+\frac{1}{(1+k^2)}+\frac{V^2}{\omega^2 r_0^2}}}. \tag{55}$$

Obviously, the latter equation contains the following two dimensionless quantities

$$\alpha_1 \equiv \frac{V}{\omega r_0}, \tag{56}$$

$$\alpha_2 \equiv |k|. \tag{57}$$

For different limits of α_1 and α_2, the relative importance of the solar wind effect on along-field diffusion, ζ, can be discussed. The results are summarized in Table 1.

4.2.1. *The condition* $\alpha_1 \gg \alpha_2 \gg 1$

Here, we suppose

$$\alpha_1 \gg \alpha_2 \gg 1, \tag{58}$$

which contains the inequalities

$$\alpha_1 \gg \alpha_2, \tag{59}$$

$$\alpha_1 \gg 1, \tag{60}$$

$$\alpha_2 \gg 1. \tag{61}$$

For the condition (58), Equation (55) is simplified as

$$\zeta = |k|Vr_0. \tag{62}$$

It is known that particle diffusion coefficient κ_{zz} can be written as

$$\kappa_{zz} = \frac{u\lambda}{3}, \tag{63}$$

where λ is the mean free path of particles. With Equations (62) and (63), the relative importance of solar wind effect, ζ, on the along-field diffusion is shown as follows

$$\frac{\zeta}{\kappa_{zz}} = 3|k|\frac{Vr_0}{u\lambda}. \tag{64}$$

It is obvious that the following dimensionless quantity

$$\beta_1 = |k|\frac{Vr_0}{u\lambda} \tag{65}$$

determines the relative importance of ζ to κ_{zz}. If $\beta_1 \gtrsim 1$, the influence of solar wind effect, ζ, on the along-field diffusion should be taken into account, conversely, if $\beta_1 \ll 1$, ζ can be ignored. For the typical values of V, u, and λ in Section 3, the dimensionless quantity becomes

$$\beta_1 = |k|r_0\frac{V}{u\lambda} \sim 10|k|r_0. \tag{66}$$

Thus, we only need to explore the value of kr_0. The solar magnetic field can be described by

$$r_0 = \frac{V}{\omega}\theta \tag{67}$$

with solar wind speed V and solar rotation speed ω. It is known that the angle θ of polar system inscreases in an clockwise direction. The parametric formulas of Equation (67) in coordinate system $x'' - y'' - z''$ are shown as

$$x'' = \frac{V}{\omega}\theta \cos\theta, \tag{68}$$

$$y'' = \frac{V}{\omega}\theta \sin\theta. \tag{69}$$

The slope of the tangent line at the point A is $k' = \tan\phi$ with the angle ϕ between x''-axis and the tangent. With the latter equations, the tangent slope of point A on the magnetic field line in coordinate system $x'' - y'' - z''$ can be written as

$$k' = \tan\phi = \frac{dy''}{dx''} = \frac{dy''/d\theta}{dx''/d\theta} = \frac{\theta + \tan\theta}{1 - \theta\tan\theta}. \tag{70}$$

Similarly, the slope k of the tangent is $k = \tan\varphi$ in coordinate system $x' - y' - z'$, which satisfies

$$k = \tan\varphi = \tan\left(\phi + \theta - \frac{\pi}{2}\right) = \frac{2\theta\tan\theta + \tan^2\theta - 1}{\theta + 2\tan\theta - \theta\tan^2\theta}. \tag{71}$$

It is easily proved that slope k is the generally increasing function of variable θ. It is suggested that the points with $\theta = n\pi + \pi/2$ for integer numbers n in Equation (71) are removable singularities, which do not

affect the monotonicity of function k with θ. Therefore, the quantity k is an appropriate parameter to reflect the influence of solar wind on particle transport, ζ, at different spatial locations within the heliosphere. Accordingly, Equation (71) becomes

$$|k|r_0 = \frac{V}{\omega}\theta\left|\frac{2\theta\tan\theta + \tan^2\theta - 1}{\theta + 2\tan\theta - \theta\tan^2\theta}\right|. \tag{72}$$

If $\theta \to 0$, for typical parameter values V and ω listed above, we can obtain

$$|k|r_0 \to 10^{10}\text{m}. \tag{73}$$

Therefore, with Equation (66), we can find

$$\beta_1 \sim 10|k|r_0 \sim 10^{11} \gg 1. \tag{74}$$

In addition, Inequality (60) denotes $r_0 \ll V/\omega \sim 10^{11}\text{m} \sim 1$ AU, which indicates that the point A is located in the inner heliosphere and Inequality (61). It is noted that Inequality (61), is consistent with the condition, Inequality (58). However, Inequality (58) leads to the relation $\alpha_1/\alpha_2 \gg 1$, i.e., $|k|r_0 \ll V/\omega$, which is contradictory to Equation (72) in the inner heliosphere. Therefore, for this case, the influence of solar wind on the along-field diffusion, ζ, may not be taken into account.

4.2.2. *The condition $\alpha_2 \gg \alpha_1 \gg 1$*

In this subsection, we suppose

$$\alpha_2 \gg \alpha_1 \gg 1, \tag{75}$$

which contains

$$\alpha_2 \gg \alpha_1, \tag{76}$$

$$\alpha_2 \gg 1, \tag{77}$$

$$\alpha_1 \gg 1. \tag{78}$$

Equation (55) becomes

$$\zeta = -10\frac{V^2}{\omega}\ln\frac{\alpha_2}{\alpha_1}. \tag{79}$$

With the along-field diffusion coefficient formula, i.e., Equation (63), we find

$$\frac{\zeta}{\kappa_{zz}} = -30\frac{V^2}{u\omega\lambda}\ln\frac{\alpha_2}{\alpha_1}. \tag{80}$$

Obviously, the latter equation is determined by the following two dimensionless quantities

$$\beta_2 = \frac{V^2}{u\omega\lambda}, \tag{81}$$

$$\beta_3 = \frac{\alpha_2}{\alpha_1} = \frac{\omega r_0|k|}{V}. \tag{82}$$

As shown in subsection 4.2.1, the condition $\alpha_1 \gg 1$ and $\alpha_2 \gg 1$ can only be satisfied in the inner heliosphere. However, from Inequality (76) we can obtain $\alpha_2/\alpha_1 = \beta_3 \gg 1$. For typical parameter values V and ω in subsection 4.2.1, we find

$$|k|r_0 \gg \frac{V}{\omega}, \tag{83}$$

which cannot be satisfied in the inner heliosphere. Therefore, in this condition, the solar wind effect on along-field diffusion of energetic particles, ζ, might be ignored.

4.2.3. *The condition $\alpha_1 \sim \alpha_2 \gg 1$*

In the following, we suppose

$$\alpha_1 \sim \alpha_2 \gg 1, \tag{84}$$

which contains two conditions

$$\alpha_1 \sim \alpha_2, \tag{85}$$

$$\alpha_1 \gg 1, \tag{86}$$

$$\alpha_2 \gg 1. \tag{87}$$

The condition (86) denotes $|k|r_0 \sim V/\omega$, which is consistent with Equation (72) at least in the inner heliosphere. In addition, Inequality (86) corresponds to $r_0 \ll V/\omega \sim 10^{11}m$, which indicates the point A is located in the inner heliosphere. Inequality (87) also shows that the point A is close to the sun.

Additionly, for the condition (84), Equation (55) is simplified as

$$\zeta = 0.4 \frac{V^2}{\omega}. \tag{88}$$

The ratio of A to κ_{zz} is

$$\frac{\zeta}{\kappa_{zz}} = 1.2 \frac{V^2}{u\omega\lambda}, \tag{89}$$

which shows that the following dimensionless quantity determines the relative importance of solar wind effect

$$\beta_2 = \frac{V^2}{u\omega\lambda}. \tag{90}$$

When $\beta_2 \sim 1$, the solar wind effect on along-field diffusion of energetic particles, ζ, is important. For the typical parameter values V, ω, u, and λ listed in subsection 4.2.1, we can obtain $\beta_2 = 0.3$. In summary, for Inequality (84), i.e., the point A located in the inner heliosphere, solar wind effect on along-field diffusion, ζ, is relatively important, so it should be considered.

4.2.4. *The condition $\alpha_1 \gg 1$ and $\alpha_2 \ll 1$*

Here, we suppose

$$\alpha_1 \gg 1, \tag{91}$$

and

$$\alpha_2 \ll 1. \tag{92}$$

For Inequalities (91) and (92), Equation (55) is simplified as

$$\zeta = |k| V r_0, \tag{93}$$

which is identical to Equation (62). Accordingly, the dimensionless quantity controlling the relative importance is

$$\beta_1 = |k| \frac{V r_0}{u\lambda}. \tag{94}$$

In addition, Inequality (91), i.e., $r_0 \ll V/\omega$, corresponds to $r_0 \ll 10^{11} m$, which denotes that the point A is in the inner heliosphere. However, Inequality (92) represents that the slope of tangent line is very small, which indicates that the point A is located in the outer heliosphere. Thus, the two Inequalities (91) and (92) are mutually contraditory. Therefore, for this case, the solar wind effect, ζ, might not be considered.

4.2.5. *The condition $\alpha_2 \gg 1$ and $\alpha_1 \ll 1$*

In this subsection, we suppose the two inequalities

$$\alpha_2 \gg 1, \tag{95}$$

$$\alpha_1 \ll 1. \tag{96}$$

Using the the latter two inequalities, from Equation (55) we can obtain

$$\zeta = 10 \frac{V^2}{\omega}. \tag{97}$$

Comparing the latter formula with κ_{zz}, we can find the following dimensionless quantities

$$\beta_2 = \frac{V^2}{u\omega\lambda}. \tag{98}$$

Inequality (95) indicates the point A is in the inner heliosphere. Additionly, for the typical parameter values listed in Section 3, Inequality (96) denotes $r_0 \gg 10^{11}$ m ~ 1 AU which indicates that the point A is located in the outer heliosphere. Thus, Inequatity (95) is contradictory with Inequality (96). Therefore, for this case the solar wind effect, ζ, on energetic particle transport may not be considered.

4.2.6. *The condition $\alpha_1 \ll 1$ and $\alpha_2 \ll 1$*

Now, we suppose the two inequalities

$$\alpha_1 \ll 1, \tag{99}$$

$$\alpha_2 \ll 1. \tag{100}$$

For the typical parameter values listed in Subsection 4.2.1, Inequality (99) gives $r_0 \gg 10^{11} m$, which indicates the point A located in the outer heliosphere. This is consistent with Inequality (100).

In addition, for the two Inequalitites (99) and (100), Equation (55) becomes

$$\zeta = |k|\frac{V^2}{\omega}. \tag{101}$$

By comparing the latter formula with κ_{zz}, we obtain

$$\frac{\zeta}{\kappa_{zz}} = 3\frac{|k|V^2}{u\omega\lambda}, \tag{102}$$

which is determined by the dimensionless quantities

$$\beta_4 = \frac{|k|V^2}{u\omega\lambda}. \tag{103}$$

For the parameter values V, u, ω, and λ listed in Section 3, from Equation (103) we can obtain $|k| \geq 3$, which is contradictory to Inequality (100). Therefore, when energetic particle transport in the outer helioshere, the solar wind effect, ζ, might not be taken into account.

4.3. *Relative importance of the solar wind and adiabatic focusing effects*

The second term on the right-hand side of Equation (29) was evaluated by Wang & Qin (2018, 2019) as

$$T_2 = \kappa_z\kappa_{tz} \approx \frac{14}{135}\frac{u\lambda^3}{L^2}. \tag{104}$$

Acccordingly, the ratio of adiabatic focusing effect term T_2 to along-field Fick's law diffusion term T_1 is

$$\frac{T_2}{T_1} = \frac{\kappa_z\kappa_{tz}}{\kappa_{zz}} \approx 0.3\xi^2. \tag{105}$$

with the dimensionless quantity

$$\xi = \frac{\lambda}{L}. \tag{106}$$

Here, λ is the mean free path of charged energetic particles and L is the characteristic length of adiabatic focusing. Obviously, the dimensionless quantity ξ determines the relative importance of T_2 to T_1. From Equations (105) and (106), we can find that if $\xi \gtrsim 1$, adiabatic focusing effect need to be taken into account. If $\xi \ll 1$, the adiabatic focusing effect should be ignored.

In Subsection 4.2, the influence of solar wind on along-field diffusion is explored. In the condition with strong solar wind effect, $\alpha_1 \sim \alpha_2 \gg 1$, considering the formulas of the solar wind effect T_3 and adiabatic focusing effect T_2, we have

$$\frac{T_3}{T_2} = \frac{27}{7} \frac{V^2 L^2}{u\omega\lambda^3}. \tag{107}$$

Obviously, the following dimensionless quantity can be found

$$\gamma = \frac{V^2 L^2}{u\omega\lambda^3} = \frac{V^2}{u\omega\lambda}\xi^2. \tag{108}$$

For typical parameter values in Section 3, thedimensionless quantity $\gamma \approx 0.3$. Thus, if $\xi \sim 1$ is satisfied, both the solar wind effect and the adiabatic focusing effect should be considered.

5. THE INFLUENCE OF SOLAR WIND EFFECT ON CROSS-FIELD DIFFUSION

Next, based on Equation (43), we investigate the solar wind effect on eneregtic particle cross-field diffusion.

5.1. *The integrating form of the solar wind effect, η*

From Equation (43), the solar wind effect on the cross-field diffusion can be written as

$$\eta = \int_{-x_0}^{x_0} \mathrm{d}x x V_x F_b - \int_{-x_0}^{x_0} \mathrm{d}x x F_b \int_{-x_0}^{x_0} \mathrm{d}x V_x F_b \tag{109}$$

From Figure 1, the following formula can be found

$$V_\perp = V \sin \psi. \tag{110}$$

In this article, for the sake of simplicity, we assume that $V_\perp \sim V_x$ and the solar wind speed V is constant. Thus, Equation (109) becomes

$$\eta = V \int_{-x_0}^{x_0} \mathrm{d}x x F_b \sin \psi - V \int_{-x_0}^{x_0} \mathrm{d}x x F_b \int_{-x_0}^{x_0} \mathrm{d}x F_b \sin \psi. \tag{111}$$

Obviously, the angle ψ varies depending on the point A and obeys the following formula on the ecliptic plane (e.g., Qin & Wu 2021)

$$\sin \psi = \frac{\omega r}{V} \left(\frac{\omega^2 r^2}{V^2} + 1 \right)^{-\frac{1}{2}}. \tag{112}$$

Inserting the latter formula into Equation (111), we have

$$\eta = V \int_{-x_0}^{x_0} \mathrm{d}x \, x F_b \frac{\omega r}{V} \left(\frac{\omega^2 r^2}{V^2} + 1 \right)^{-\frac{1}{2}} - V \int_{-x_0}^{x_0} \mathrm{d}x \, x F_b \int_{-x_0}^{x_0} \mathrm{d}x F_b \frac{\omega r}{V} \left(\frac{\omega^2 r^2}{V^2} + 1 \right)^{-\frac{1}{2}}. \tag{113}$$

To proceed, for simplicity, we assume that the distribution function F_b is approximately constant. Thus, the formula $\int_{-x_0}^{x_0} \mathrm{d}x \, x F_b = 0$ holds, and Equation (113) becomes

$$\eta = V \int_{-x_0}^{x_0} \mathrm{d}x \, x F_b \frac{\omega r}{V} \left(\frac{\omega^2 r^2}{V^2} + 1 \right)^{-\frac{1}{2}}. \tag{114}$$

From the equation of normal line, $y' = -x'/k + r_0$, we can obtain the formulas $\mathrm{d}x = \sqrt{1 + (\mathrm{d}y'/\mathrm{d}x')^2}\mathrm{d}x' = \sqrt{1 + 1/k^2}\mathrm{d}x'$ and $x = \sqrt{1 + (\mathrm{d}'y/\mathrm{d}x')^2}x' = \sqrt{1 + 1/k^2}x'$, Equation (114) becomes

$$\eta = V \left(1 + \frac{1}{k^2} \right) \int_{-0.1r_0}^{0.1r_0} \mathrm{d}x' \, x' F_b \frac{\omega r}{V} \left(\frac{\omega^2 r^2}{V^2} + 1 \right)^{-\frac{1}{2}}, \tag{115}$$

where we set $x_0 = 0.1r_0/\cos\psi$. Thus, from the normalization condition $\int_{-0.1r_0}^{0.1r_0} \mathrm{d}x' F_b = 1$, we can find

$$F_b = \frac{5}{r_0}. \tag{116}$$

Accordingly, Equation (115) becomes

$$\eta = V \frac{5}{r_0} \left(1 + \frac{1}{k^2} \right) \int_{-0.1r_0}^{0.1r_0} \mathrm{d}x' \, x' \frac{\omega r}{V} \left(\frac{\omega^2 r^2}{V^2} + 1 \right)^{-\frac{1}{2}}. \tag{117}$$

Using the dimensionless quantity $\alpha_1 = V/(\omega r_0)$, we can rewrite Equation (115) as

$$\eta = V \frac{5}{r_0} \left(1 + \frac{1}{k^2} \right) \int_{-0.1r_0}^{0.1r_0} \mathrm{d}x' \, x' \left(\alpha_1^2 \frac{r_0^2}{r^2} + 1 \right)^{-\frac{1}{2}}. \tag{118}$$

The latter equation is very complex, so it is not easy to be evaluated. However, the relation $r_0/r \sim 1$ approximately holds because the integral interval is small enough, so that Equation (118) can be qualitatively explored for $\alpha_1 \gg 1$ and $\alpha_1 \ll 1$.

5.2. *The condition $\alpha_1 \gg 1$*

For $\alpha_1 \gg 1$, with $r = \sqrt{x'^2 + (-x'/k + r_0)^2}$, Equation (118) is simplified as

$$\eta = 10 \frac{\omega}{r_0} \frac{1 + k^2}{k^2} \int_0^{0.1r_0} \mathrm{d}x' \, x' \sqrt{x'^2 + (-x'/k + r_0)^2}. \tag{119}$$

After a lengthy mathematical performance, the latter equation becomes

$$\eta = \frac{10}{3}\omega r_0^2 \frac{\left(1+k^2\right)^{3/2}}{k^3}\left\{\left[\left(0.1 - \frac{k}{1+k^2}\right)^2 + \frac{k^4}{(1+k^2)^2}\right]^{3/2} - \left[\frac{k^2}{(1+k^2)^2} + \frac{k^4}{(1+k^2)^2}\right]^{3/2}\right\}. \qquad (120)$$

In the following, we evaluate the solar wind effect on cross-field diffusion, η, for $\alpha_2 = |k| \to \infty$ and $\alpha_2 = |k| \to 0$, respectively.

5.2.1. *The condition $\alpha_2 = |k| \to \infty$*

For the condition

$$\alpha_2 = |k| \to \infty, \qquad (121)$$

we can easily obtain

$$\eta \approx 1.5\omega r_0^2. \qquad (122)$$

Comparing the latter formula with $\kappa_{xx} = u\lambda_\perp/3$ we obtain

$$\frac{\eta}{\kappa_{xx}} = 4.5\frac{\omega r_0^2}{u\lambda_\perp} \sim \frac{\omega r_0^2}{u\lambda_\perp}. \qquad (123)$$

Obviously, the following dimensionless quantity determines the relative importance of solar wind effect on cross-field diffusion

$$\delta_1 = \frac{\omega r_0^2}{u\lambda_\perp}. \qquad (124)$$

In the condition (121), the point A should be close to the sun. Thus, the distance r_0 is much less than 1 AU, i.e., $r_0 \ll 10^{11}$ m. In addition, for the typical parameter values ω, u, and $\lambda_\perp$ in Section 3, we have

$$\delta_1 \ll 1, \qquad (125)$$

which indicates the solar wind effect on the along-field diffusion, η, may be ignored.

5.2.2. *The condition $\alpha_2 \to 0$*

For $\alpha_2 \to 0$, we can obtain

$$\eta \approx 0.1\frac{\omega r_0^2}{k^3}. \qquad (126)$$

Comparing the term η and the cross-field diffusion coefficient κ_{xx} yields

$$\frac{\eta}{\kappa_{xx}} = 0.3 \frac{\omega r_0^2}{u \lambda_\perp k^3}. \tag{127}$$

It is obvious that the following dimensionless quantity determines the relative importance of solar wind to cross-field diffusion

$$\delta_2 = \frac{\omega r_0^2}{u \lambda_\perp k^3}. \tag{128}$$

For the typical parameter values u, ω, $\lambda_\perp$ in Section 3, we can obtain

$$\delta_2 \sim \left(10^{-11} r_0\right)^2 \frac{1}{k^3}. \tag{129}$$

For $\delta_2 \geq 1$, the relation $k \leq \left(10^{-11} r_0\right)^{2/3}$ holds. However, the condition $\alpha_1 \gg 1$ corresponds to $r_0 \ll V/\omega \sim 10^{11}$ m, which denotes that the point A is in the inner heliosphere. The condition $\alpha_1 \gg 1$ is contradictory to $\alpha_2 \to 0$ which indicates the point A in the outer heliosphere. Therefore, the solar wind effect on cross-field diffusion, η, may be ignored.

5.3. The condition $\alpha_1 \ll 1$

If $\alpha_1 \ll 1$, with $\int_{-0.1r_0}^{0.1r_0} dx F_b = 1$, Equation (113) becomes

$$\eta = V \int_{-0.1r_0}^{0.1r_0} dx x F_b - V \int_{-0.1r_0}^{0.1r_0} dx x F_b = 0. \tag{130}$$

Obviously, for this case, the solar wind effect, η, might not be taken into account.

6. SUMMARY AND CONCLUSION

In this paper, starting from the focusing equation, we derive the formulas of the mean square displacement definitions of along- and cross-field diffusion coefficients, κ_{zz}^{DV} and κ_{xx}^{DV}, respectively. It is demonstrated that κ_{zz}^{DV} includes solar wind effect, i.e., $\zeta = \langle zV_z \rangle - \langle z \rangle \langle V_z \rangle$, and κ_{xx}^{DV} contains $\eta = \langle xV_x \rangle - \langle x \rangle \langle V_x \rangle$.

For different limits of dimensionless quantities $\alpha_1 = V/(\omega r_0)$ and $\alpha_2 = |k|$, the relative importance of the solar wind effect on along-field diffusion, ζ, is expored, and corresponding dimensionless quantities are obtained, which are summarized in Table 1. For the condition $\alpha_1 \sim \alpha_2 \gg 1$, we find that when the point A

is close to the sun, the relative importance of solar wind effect on along-field diffusion, ζ, should be taken into account. In this condition, we find that when $\xi = \lambda/L \geq 1$, the adiabatic focusing effects need also to be considered. Next, the relative importance of solar wind effect on cross-field diffusion, η, is investigated in several extreme conditions, we find that the solar wind effect on cross-field diffusion, η, might be ignored.

The results obtained in this paper have certain significance in the transport of energetic particles not only in the heliosphere, but also in many other scenarios, such as, planetary magnetoshere and ionoshere, intersteller space, the spaces close to neutron stars, supernova remanents and so on. It is possible that in some conditions, the solar wind effect on energetic particle diffusion in the heliosphere is not important, but the background plasma speed effect on energetic particle diffusion in other scenarios is not ignorable.

In this article, the results we get are not very conclusive. We only perform the exploration in some extreme conditions. In addition, we set some paramters with typical values as shown in Section 3, which are standard in 1 AU for energetic particles. However, using the typical parameter values, we discuss solar wind effect on diffusion coefficients in some special conditions, e.g., $r_0 \ll 1$ AU, in which the typical parameter values may be not appropriate in the condition. In the future, we will investigate solar wind effect on diffusion coefficients in more general conditions, using more reasonalble typical parameter values in special conditions.

Other physical effects, e.g., solar wind of spatial derivative, momentum transport, etc., on both spatial diffusion and drift coefficients, may also be investigated. In past decades, the non-diffusion, i.e., subdiffusion and superdiffusion, has gained more and more interest due to its general applications in numerous research field. We may also investigate the influence of the solar wind and adiabatic focusing on non-diffusion of energetic particles.

A. THE FOCUSING EQUATION SATISFYING PARTICLE NUMBER CONSERVATION LAW

The focusing equation, which has been broadly used to research charged particle transport, is as follow

$$\frac{\partial f}{\partial t} = \nabla \cdot (\kappa_\perp \cdot \nabla f) - \left(u\mu\hat{\boldsymbol{b}} + \boldsymbol{V} \right) \cdot \nabla f + \frac{\partial}{\partial \mu}\left(D_{\mu\mu}\frac{\partial f}{\partial \mu} \right) + \frac{dp}{dt}\frac{\partial f}{\partial p} + \frac{d\mu}{dt}\frac{\partial f}{\partial \mu} \tag{A-1}$$

with

$$\frac{dp}{dt} = p\left[\frac{1-\mu^2}{2}\left(\nabla \cdot \boldsymbol{V} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}} : \nabla \boldsymbol{V} \right) + \mu^2 \hat{\boldsymbol{b}}\hat{\boldsymbol{b}} : \nabla \boldsymbol{V} \right], \tag{A-2}$$

$$\frac{d\mu}{dt} = \frac{1-\mu^2}{2}\left[-\frac{u}{L} - \mu\left(\nabla \cdot \boldsymbol{V} - 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} : \nabla \boldsymbol{V} \right) \right]. \tag{A-3}$$

Here, $L = \left(\hat{\boldsymbol{b}} \cdot \nabla \ln B \right)^{-1}$. However, this formulation does not satisfy particle number conservation law which is one of the most important physical laws. Therefore, we have to find a focusing equation which satisfies the particle number conservation law.

For spherical system (p, θ, ϕ), the nabla operator of momentum space is

$$\nabla_p f = \frac{\partial f}{\partial p}\boldsymbol{e}_p + \frac{1}{p}\frac{\partial f}{\partial \theta}\boldsymbol{e}_\theta + \frac{1}{p\sin\theta}\frac{\partial f}{\partial \phi}\boldsymbol{e}_\phi. \tag{A-4}$$

In this paper, we only consider gyrotropic case, and Equation (A-4) becomes

$$\nabla_p f = \frac{\partial f}{\partial p}\boldsymbol{e}_p + \frac{1}{p}\frac{\partial f}{\partial \theta}\boldsymbol{e}_\theta. \tag{A-5}$$

Because $\mu = \cos\theta$, we have

$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial \mu}\frac{\partial \mu}{\partial \theta} = -\sin\theta\frac{\partial}{\partial \mu}. \tag{A-6}$$

Inserting the latter equation into Equation (A-5) gives

$$\nabla_p f = \frac{\partial f}{\partial p}\boldsymbol{e}_p - \sin\theta\frac{1}{p}\frac{\partial f}{\partial \mu}\boldsymbol{e}_\theta. \tag{A-7}$$

To proceed, we have

$$\boldsymbol{a} \cdot \nabla_p f = \left(a_p\boldsymbol{e}_p + a_\theta\boldsymbol{e}_\theta \right) \cdot \left(\frac{\partial f}{\partial p}\boldsymbol{e}_p - \frac{\sin\theta}{p}\frac{\partial f}{\partial \mu}\boldsymbol{e}_\theta \right) = a_p\frac{\partial f}{\partial p} - a_\theta\frac{\sin\theta}{p}\frac{\partial f}{\partial \mu} \tag{A-8}$$

With the following formula

$$\boldsymbol{a} \cdot \nabla_p f = \frac{dp}{dt}\frac{\partial f}{\partial p} + \frac{d\mu}{dt}\frac{\partial f}{\partial \mu}, \tag{A-9}$$

we obtain

$$\frac{dp}{dt} = a_p, \tag{A-10}$$

$$\frac{d\mu}{dt} = -a_\theta \frac{\sin\theta}{p}. \tag{A-11}$$

That is, the latter formulas can be rewritten as

$$a_p = \frac{dp}{dt}, \tag{A-12}$$

$$a_\theta = -\frac{d\mu}{dt}\frac{p}{\sin\theta}. \tag{A-13}$$

Considering Equations (A-2) and (A-3), we can find that

$$a_p = p\left[\frac{1-\mu^2}{2}\left(\nabla\cdot V - \hat{b}\hat{b} : \nabla V\right) + \mu^2\hat{b}\hat{b} : \nabla V\right], \tag{A-14}$$

$$a_\theta = \frac{p}{\sin\theta}\frac{1-\mu^2}{2}\left[\frac{u}{L} + \mu\left(\nabla\cdot V - 3\hat{b}\hat{b} : \nabla V\right)\right]. \tag{A-15}$$

Inserting formulas (A-12) and (A-13) into the following equation

$$\nabla_p \cdot (af) = \frac{1}{p^2}\frac{\partial}{\partial p}\left(p^2 a_p f\right) - \frac{1}{p}\frac{\partial}{\partial\mu}\left(\sin\theta a_\theta f\right), \tag{A-16}$$

we can obtain

$$\nabla_p \cdot (af) = \frac{1}{p^2}\frac{\partial}{\partial p}\left\{p^3\left[\frac{1-\mu^2}{2}\left(\nabla\cdot V - \hat{b}\hat{b} : \nabla V\right) + \mu^2\hat{b}\hat{b} : \nabla V\right]f\right\}$$
$$+ \frac{\partial}{\partial\mu}\left\{\frac{1-\mu^2}{2}\left[-\frac{u}{L} - \mu\left(\nabla\cdot V - 3\hat{b}\hat{b} : \nabla V\right)\right]f\right\}. \tag{A-17}$$

The Fokker-Planck equation is shown as

$$\frac{\partial f}{\partial t} + \nabla\cdot(vf) + \nabla_p\cdot(af) = \nabla\cdot(\kappa\cdot\nabla f) + \nabla_p\cdot\left(\kappa\cdot\nabla_p f\right), \tag{A-18}$$

which satisfies the particle number conservation law. By considering the latter formula and the following fomulas

$$v = u\mu\hat{b} + V, \tag{A-19}$$

$$\nabla\cdot(\kappa\cdot\nabla f) = \nabla\cdot(\kappa_\perp\cdot\nabla f), \tag{A-20}$$

$$\nabla_p\cdot\left(\kappa\cdot\nabla_p f\right) = \frac{\partial}{\partial\mu}\left(D_{\mu\mu}\frac{\partial f}{\partial\mu}\right), \tag{A-21}$$

Equation (A-18) becomes

$$
\begin{aligned}
\frac{\partial f}{\partial t} &= \nabla \cdot (\kappa_\perp \cdot \nabla f) - \nabla \cdot \left[\left(u\mu\hat{\boldsymbol{b}} + \boldsymbol{V}\right) f\right] + \frac{\partial}{\partial \mu}\left(D_{\mu\mu}\frac{\partial f}{\partial \mu}\right) \\
&+ \frac{1}{p^2}\frac{\partial}{\partial p}\left\{p^3\left[\frac{1-\mu^2}{2}\left(\nabla \cdot \boldsymbol{V} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}} : \nabla\boldsymbol{V}\right) + \mu^2\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} : \nabla\boldsymbol{V}\right]f\right\} \\
&+ \frac{\partial}{\partial \mu}\left\{\frac{1-\mu^2}{2}\left[-\frac{u}{L} - \mu\left(\nabla \cdot \boldsymbol{V} - 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} : \nabla\boldsymbol{V}\right)\right]f\right\},
\end{aligned}
\tag{A-22}
$$

which is our starting point of this paper.

In fact, if the incompressible condition

$$
\nabla \cdot \boldsymbol{v} + \nabla_p \cdot \boldsymbol{a} = 0
\tag{A-23}
$$

holds, Equation (A-1) can also satisfy particle number conservation. Now, the specific form of the incompressible condition is derived. With the divergence formula

$$
\nabla \cdot \boldsymbol{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta f_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial f_\phi}{\partial \phi},
\tag{A-24}
$$

for gyrotropic case the latter formula becomes

$$
\nabla \cdot \boldsymbol{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta f_\theta\right)
\tag{A-25}
$$

Using $\mu = \cos\theta$, the latter formula can be rewritten as

$$
\nabla \cdot \boldsymbol{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f_r\right) - \frac{1}{r}\frac{\partial}{\partial \mu}\left(\sin\theta f_\theta\right).
\tag{A-26}
$$

In addition, the divergence formula for velocity is shown as

$$
\nabla \cdot \boldsymbol{v} = \nabla \cdot \left(u\mu\hat{\boldsymbol{b}} + \boldsymbol{V}\right) = \nabla \cdot \boldsymbol{V}.
\tag{A-27}
$$

Similarly, with Equations (A-12) and (A-13), the divergence formula for acceleration can be obtained

$$
\begin{aligned}
\nabla_p \cdot \boldsymbol{a} &= \frac{1}{p^2}\frac{\partial}{\partial p}\left(p^2 a_p\right) - \frac{1}{p}\frac{\partial}{\partial \mu}\left(\sin\theta a_\theta\right) \\
&= \frac{1}{p^2}\frac{\partial}{\partial p}\left\{p^3\left[\frac{1-\mu^2}{2}\left(\nabla \cdot \boldsymbol{V} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}} : \nabla\boldsymbol{V}\right) + \mu^2\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} : \nabla\boldsymbol{V}\right]\right\} \\
&- \frac{1}{p}\frac{\partial}{\partial \mu}\left\{\sin\theta\frac{p}{\sin\theta}\frac{1-\mu^2}{2}\left[\frac{u}{L} + \mu\left(\nabla \cdot \boldsymbol{V} - 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} : \nabla\boldsymbol{V}\right)\right]\right\}
\end{aligned}
\tag{A-28}
$$

To continue, the latter formula can be simplified as

$$\nabla_p \cdot \boldsymbol{a} = \nabla \cdot \boldsymbol{V} + \left(3\mu^2 - 1\right)\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} : \nabla \boldsymbol{V} + \frac{u\mu}{L}. \tag{A-29}$$

Combining Equations (A-27) and (A-29) yields

$$\nabla \cdot \boldsymbol{v} + \nabla_p \cdot \boldsymbol{a} = 2\nabla \cdot \boldsymbol{V} + \left(3\mu^2 - 1\right)\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} : \nabla \boldsymbol{V} + \frac{u\mu}{L} = 0 \tag{A-30}$$

Thus, the condition of particle number conservation can be rewritten as

$$-\frac{u\mu}{L} = 2\nabla \cdot \boldsymbol{V} + \left(3\mu^2 - 1\right)\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} : \nabla \boldsymbol{V} \tag{A-31}$$

If the latter condition is satisfied, Equation (A-1) also satisfies the particle number conservation law.

B. THE FOCUSING EQUATION WITHOUT TENSOR OPERATION

For mathematical tractability, the focusing equation can be rewritten as

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \nabla \cdot (\kappa_\perp \cdot \nabla f) - \nabla \cdot \left[\left(u\mu\hat{\boldsymbol{b}} + \boldsymbol{V}\right)f\right] + \frac{\partial}{\partial \mu}\left(D_{\mu\mu}\frac{\partial f}{\partial \mu}\right) \\
&+ \frac{1}{p^2}\frac{\partial}{\partial p}\left\{p^3\left[\frac{1-\mu^2}{2}\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y}\right) + \mu^2\frac{\partial V_z}{\partial z}\right]f\right\} \\
&+ \frac{\partial}{\partial \mu}\left\{\frac{1-\mu^2}{2}\left[-\frac{u}{L} - \mu\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} - 2\frac{\partial V_z}{\partial z}\right)\right]f\right\},
\end{aligned} \tag{B-1}$$

with the following formulas

$$\boldsymbol{V}^{SW} = V_z^{SW}\hat{\boldsymbol{b}} + V_x^{SW}\hat{\boldsymbol{n}}_x + V_y^{SW}\hat{\boldsymbol{n}}_y, \tag{B-2}$$

$$\nabla = \frac{\partial}{\partial z}\hat{\boldsymbol{b}} + \frac{\partial}{\partial x}\hat{\boldsymbol{n}}_x + \frac{\partial}{\partial y}\hat{\boldsymbol{n}}_y, \tag{B-3}$$

$$\hat{\boldsymbol{b}} \cdot \nabla = \hat{\boldsymbol{b}} \cdot \left(\frac{\partial}{\partial z}\hat{\boldsymbol{b}} + \frac{\partial}{\partial x}\hat{\boldsymbol{n}}_x + \frac{\partial}{\partial y}\hat{\boldsymbol{n}}_y\right) = \frac{\partial}{\partial z}, \tag{B-4}$$

$$\nabla \cdot \hat{\boldsymbol{b}} = \left(\frac{\partial}{\partial z}\hat{\boldsymbol{b}} + \frac{\partial}{\partial x}\hat{\boldsymbol{n}}_x + \frac{\partial}{\partial y}\hat{\boldsymbol{n}}_y\right) \cdot \hat{\boldsymbol{b}} = \frac{\partial}{\partial z}, \tag{B-5}$$

$$\hat{\boldsymbol{b}} \cdot \boldsymbol{V}^{SW} = \hat{\boldsymbol{b}} \cdot \left(V_z^{SW}\hat{\boldsymbol{b}} + V_x^{SW}\hat{\boldsymbol{n}}_x + V_y^{SW}\hat{\boldsymbol{n}}_y\right) = V_z^{SW}, \tag{B-6}$$

$$\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} : \nabla \boldsymbol{V}^{SW} = \left(\hat{\boldsymbol{b}} \cdot \nabla\right)\left(\hat{\boldsymbol{b}} \cdot \boldsymbol{V}^{SW}\right) = \frac{\partial V_z^{SW}}{\partial z}, \tag{B-7}$$

$$\nabla \cdot \boldsymbol{V}^{SW} = \left(\frac{\partial}{\partial z}\hat{\boldsymbol{b}} + \frac{\partial}{\partial x}\hat{\boldsymbol{n}}_x + \frac{\partial}{\partial y}\hat{\boldsymbol{n}}_y\right)\left(V_z^{SW}\hat{\boldsymbol{b}} + V_x^{SW}\hat{\boldsymbol{n}}_x + V_y^{SW}\hat{\boldsymbol{n}}_y\right) = \frac{\partial V_z^{SW}}{\partial z} + \frac{\partial V_x^{SW}}{\partial x} + \frac{\partial V_y^{SW}}{\partial y}, \tag{B-8}$$

$$\nabla \cdot \boldsymbol{V}^{SW} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}} : \nabla \boldsymbol{V}^{SW} = \frac{\partial V_x^{SW}}{\partial x} + \frac{\partial V_y^{SW}}{\partial y}, \tag{B-9}$$

$$\nabla \cdot \left(\boldsymbol{V}^{SW} F \right) = \left(\frac{\partial}{\partial z} \hat{\boldsymbol{b}} + \frac{\partial}{\partial x} \hat{\boldsymbol{n}}_x + \frac{\partial}{\partial y} \hat{\boldsymbol{n}}_y \right) \left(V_z^{SW} F \hat{\boldsymbol{b}} + V_x^{SW} F \hat{\boldsymbol{n}}_x + V_y^{SW} F \hat{\boldsymbol{n}}_y \right)$$

$$= \frac{\partial \left(V_z^{SW} F \right)}{\partial z} + \frac{\partial \left(V_x^{SW} F \right)}{\partial x} + \frac{\partial \left(V_y^{SW} F \right)}{\partial y}, \tag{B-10}$$

$$\nabla \cdot \boldsymbol{V}^{SW} - 3\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} : \nabla \boldsymbol{V}^{SW} = \frac{\partial V_x^{SW}}{\partial x} + \frac{\partial V_y^{SW}}{\partial y} - 2\frac{\partial V_z^{SW}}{\partial z}. \tag{B-11}$$

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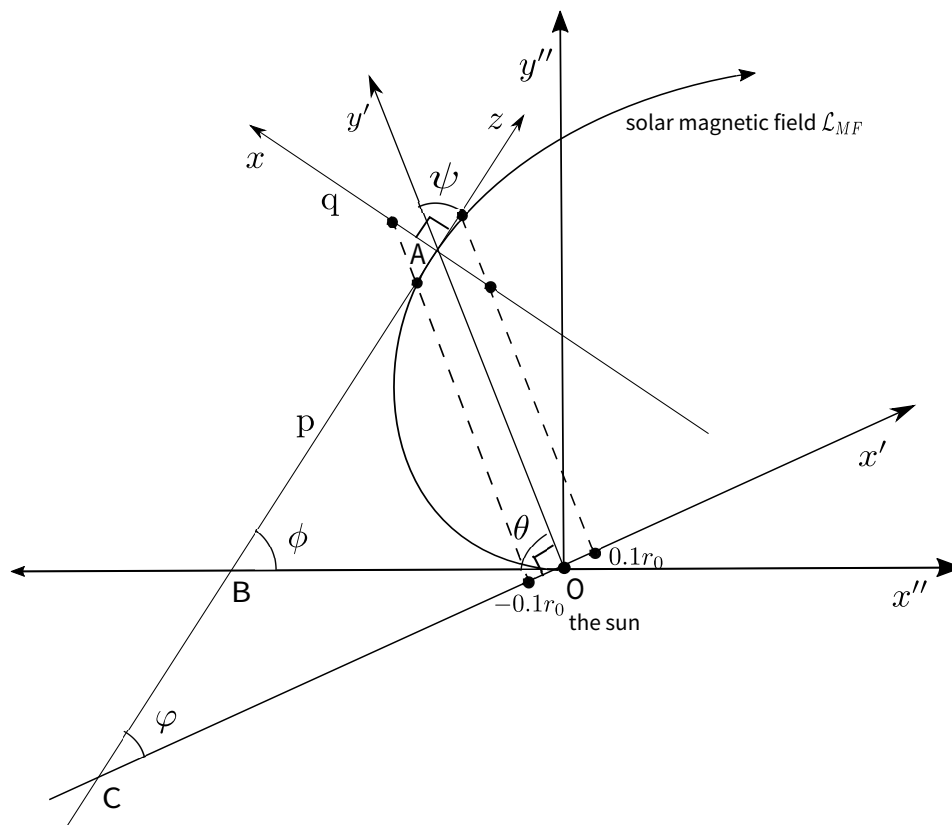Zuo, P., Zhang, M., & Rassoul, H. K. 2013, ApJ, 767, 6

Figure 1. Schematic diagram of magnetic coordinate system $x - y - z$, solar coordinate systems $x' - y' - z'$ and $x'' - y'' - z''$, the polar coordinate system, which are in ecliptic plane, and $z' \parallel z''$.

Table 1. The dimensionless quantities determining the relative importance of solar wind effect on along-field diffusion

Range	solar wind effect	
$\alpha_1 \gg \alpha_2 \gg 1$	$A = \|k\|V r_0$	$\beta_1 = \frac{\|k\|V r_0}{u\lambda}$
$\alpha_2 \gg \alpha_1 \gg 1$	$A = -10\frac{V^2}{\omega} \ln \frac{\omega r_0\|k\|}{V}$	$\beta_2 = \frac{V^2}{u\omega\lambda},$ $\beta_3 = \frac{\omega r_0\|k\|}{V}$
$\alpha_1 \sim \alpha_2 \gg 1$	$A = 0.4\frac{V^2}{\omega}$	$\beta_2 = \frac{V^2}{u\omega\lambda}$
$\alpha_1 \gg 1, \alpha_2 \ll 1$	$A = \|k\|V r_0$	$\beta_1 = \frac{\|k\|V r_0}{u\lambda}$
$\alpha_2 \ll 1, \alpha_1 \gg 1$	$A = 10\frac{V^2}{\omega}$	$\beta_2 = \frac{V^2}{u\omega\lambda}$
$\alpha_1 \ll 1, \alpha_2 \ll 1$	$A = \|k\|\frac{V^2}{\omega}$	$\beta_4 = \frac{\|k\|V^2}{u\omega\lambda}$